

ON SOME FORMALITY CRITERIA FOR DG-LIE ALGEBRAS

MARCO MANETTI

ABSTRACT. We give some formality criteria for a differential graded Lie algebra to be formal. For instance, we show that a DG-Lie algebra L is formal if and only if the natural spectral sequence computing the Chevalley-Eilenberg cohomology $H_{CE}^*(L, L)$ degenerates at E_2 .

1. INTRODUCTION

The notion of formality of a differential graded commutative algebra has been quite familiar in mathematics since the works by Deligne, Griffiths, Morgan, Sullivan [6], where it is proved that the de Rham algebra of a compact Kähler manifold X is formal and therefore its homotopy class, controlling the real homotopy type of X , is uniquely determined by the cohomology algebra $H^*(X, \mathbb{R})$.

Similarly, the notion of formality of a differential graded Lie algebra has received a great attention after the papers of Goldman, Millson [9] and Kontsevich [19]. In [9] the authors realize that the same approach of [6] can be used to prove the formality of the differential graded Lie algebra of differential forms with values in certain flat bundles of Lie algebras; as a consequence of this fact they proved that the moduli space of certain representations of the fundamental group of a compact Kähler manifold has at most quadratic singularities.

In the paper [19] Kontsevich proved that, when A is the algebra of smooth functions on a differentiable manifold, then the natural DG-Lie algebra structure on the Hochschild cohomology complex of A with coefficients in A is formal, and then proving that every finite dimensional Poisson manifold admits a canonical deformation quantization.

Recall that a DG-Lie algebra L is formal if there exists a pair of quasi-isomorphisms of DG-Lie algebras

$$L \longleftarrow M \longrightarrow H$$

with H having trivial differential. A DG-Lie algebra L is called homotopy abelian if there exists a pair of quasi-isomorphisms of DG-Lie algebras

$$L \longleftarrow M \longrightarrow H$$

with H having trivial bracket and trivial differential. Thus, a DG-Lie algebra L is homotopy abelian if and only if it is formal and the cohomology graded Lie algebra $H^*(L)$ is abelian.

In view of the general principle that, in characteristic 0 every deformation problem is controlled by a DG-Lie algebras, with quasi-isomorphic DG-Lie algebras giving the same deformation problem [9], both the notions of formality and homotopy abelianity play a central role in deformation theory. Fortunately, in literature there exist several general and very useful criteria for homotopy abelianity. For example, it is known (see e.g. [14] and references therein) that for a morphism of DG-Lie algebras $f: L \rightarrow M$ we have:

Date: December 5, 2013.

2010 Mathematics Subject Classification. 17B56, 17B70, 16T15.

Key words and phrases. Differential graded Lie algebras, spectral sequences, L_∞ -algebras.

- (1) if L is homotopy abelian and $f: H^*(L) \rightarrow H^*(M)$ is surjective, then also M is homotopy abelian;
- (2) if M is homotopy abelian and $f: H^*(L) \rightarrow H^*(M)$ is injective, then also L is homotopy abelian.

The above results, which are completely symmetric in their proofs, appear quite different in their applications, being the latter used in almost all the algebraic proofs of generalized Bogomolov-Tian-Todorov theorems [15, 18], as well in deformation theory of holomorphic Poisson manifolds and coisotropic submanifolds [3, 7].

The initial motivation for this paper was to seek for an analog of the above item (2) when the notion of homotopy abelianity is replaced with the notion of formality; it is easily verified that if M is formal, then the injectivity of $f: H^*(L) \rightarrow H^*(M)$ is not sufficient to ensure the formality of L .

In our proposed extension of item (2) for formality (Theorem 3.4), the cohomology graded Lie algebras $H^*(L)$ and $H^*(M)$ are replaced with suitable Chevalley-Eilenberg cohomology groups. In particular we shall prove that if $f: L \rightarrow M$ is a morphism of differential graded Lie algebras, with M formal and $f: H_{CE}^2(H^*(L), H^*(L)) \rightarrow H_{CE}^2(H^*(L), H^*(M))$ injective, then also L is formal.

In doing this we have been deeply inspired by the papers [17, 23], where it is explained what is the “right” obstruction to formality of a DG-algebra and by [2], where it is proved that homotopy abelianity is equivalent to the degeneration at E_1 of the natural spectral sequence computing the Chevalley-Eilenberg cohomology.

The paper is organized as follows: in Section 2 we introduce the Chevalley-Eilenberg complex of a differential graded Lie algebra as the natural generalization of the classical Chevalley-Eilenberg complex of a Lie algebra [5]; here the choice of signs of the differential is purely teleological and made by taking into account the sign convention used in the definition of décalage maps given in Section 7. Such a complex admits a natural filtration giving a cohomology spectral sequence.

Since almost all the proofs of this paper require a good knowledge of $L_\infty[1]$ -algebras and L_∞ -morphisms, for the benefit of the readers which are not familiar with these notions, in Section 3 we state the main results of the paper about formality of DG-Lie algebras. These results will be proved in Section 7 as particular cases of some more general results concerning the formality of $L_\infty[1]$ -algebras. Among the applications of these results we give a proof of the fact that for every graded vector space V the graded Lie algebra $\text{Hom}_{\mathbb{K}}^*(V, V)$ is intrinsically formal.

In Section 4 we give a short review of the definition of $L_\infty[1]$ -algebras, L_∞ -morphisms and Nijenhuis-Richardson bracket. In Section 5 we define the Chevalley-Eilenberg spectral sequence of an $L_\infty[1]$ -algebra and we prove that, for every $r > 0$, its page E_r is homotopy invariant. Finally in Section 6 we prove the formality criteria for $L_\infty[1]$ -algebras.

In the last section we show how deformation theory can be used for constructing simple examples of non formal differential graded Lie algebras.

2. THE CHEVALLEY-EILENBERG SPECTRAL SEQUENCE

Throughout this paper every vector space, tensor product, Lie algebra etc. is considered over a fixed field \mathbb{K} of characteristic 0. By a DG-vector space we shall mean a \mathbb{Z} -graded vector space equipped with a differential of degree +1; a DG-Lie algebra is a Lie object in the category of DG-vector spaces. Given a homogeneous vector v on a graded vector space, its degree will be denoted either \bar{v} or $\deg(v)$.

Let $L = (L, d, [-, -])$ be a differential graded Lie algebra and let M be an L -module. This means that $M = (M, d)$ is a DG-vector space and it is given a morphism of DG-vector spaces

$$[-, -]: M \otimes L \rightarrow M$$

such that

$$[m, [x, y]] = [[m, x], y] - (-1)^{\overline{x}\overline{y}} [[m, y], x] .$$

For instance, if $f: L \rightarrow M$ is a morphism of DG-Lie algebras, then M is an L -module via the adjoint representation $[m, x] = [m, f(x)]$.

We shall denote by $H^*(L)$ and $H^*(M)$ the cohomology of the DG-vector spaces (L, d) and (M, d) , respectively. For every integer $p \geq 0$ let's consider the DG-vector space

$$CE(L, M)^{p,*} = \text{Hom}_{\mathbb{K}}^*(L^{\wedge p}, M),$$

equipped with the natural differential $\bar{\delta}: CE(L, M)^{p,q} \rightarrow CE(L, M)^{p,q+1}$, namely

$$(\bar{\delta}\phi)(x_1, \dots, x_p) = d(\phi(x_1, \dots, x_p)) - \sum_{i=1}^p (-1)^{\bar{\phi} + \overline{x_1} + \dots + \overline{x_{i-1}}} \phi(x_1, \dots, dx_i, \dots, x_p) ,$$

where every element of $CE(L, M)^{p,*}$ is interpreted as a p -linear graded skewsymmetric map $L \times \dots \times L \rightarrow M$. As usual we intend that $L^{\wedge 0} = \mathbb{K}$ and then $CE(L, M)^{0,*} = M$.

Following [16, pag. 94], the Chevalley-Eilenberg complex of L with coefficients in M is the complex of DG-vector spaces:

$$CE(L, M): \quad 0 \rightarrow CE(L, M)^{0,*} \xrightarrow{\delta} CE(L, M)^{1,*} \xrightarrow{\delta} CE(L, M)^{2,*} \rightarrow \dots ,$$

i.e., the complex

$$CE(L, M): \quad 0 \rightarrow M \xrightarrow{\delta} \text{Hom}_{\mathbb{K}}^*(L, M) \xrightarrow{\delta} \text{Hom}_{\mathbb{K}}^*(L^{\wedge 2}, M) \rightarrow \dots$$

where the differential δ is defined in the following way:

- (1) for $m \in M$ we have $(\delta m)(x) = (-1)^{\overline{m}} [m, x]$;
- (2) for $\phi \in \text{Hom}_{\mathbb{K}}^*(L, M)$ we have

$$(\delta\phi)(x, y) = (-1)^{\bar{\phi}+1} ([\phi(x), y] - (-1)^{\overline{x}\overline{y}} [\phi(y), x] - \phi([x, y])) ;$$

- (3) for $p \geq 2$ and $\phi \in \text{Hom}_{\mathbb{K}}^*(L^{\wedge p-1}, M)$ we have:

$$\begin{aligned} (\delta\phi)(x_1, \dots, x_p) &= \\ &= (-1)^{\bar{\phi}+p-1} \left(\sum_i \chi_i [\phi(x_1, \dots, \widehat{x_i}, \dots, x_p), x_i] - \sum_{i < j} \chi_{i,j} \phi(x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_p, [x_i, x_j]) \right) \end{aligned}$$

where the $\chi_i, \chi_{i,j} \in \{\pm 1\}$ are the antisymmetric Koszul signs; when $x_1 \wedge \dots \wedge x_p \neq 0$ they are determined by the following equalities in $L^{\wedge p}$:

$$x_1 \wedge \dots \wedge x_p = \chi_i x_1 \wedge \dots \wedge \widehat{x_i} \wedge \dots \wedge x_p \wedge x_i = \chi_{i,j} x_1 \wedge \dots \wedge \widehat{x_i} \wedge \dots \wedge \widehat{x_j} \wedge \dots \wedge x_p \wedge x_i \wedge x_j .$$

The proof that $\delta^2 = 0$ may be easily reduced to the case $M = L$: in fact, denoting by $K = L \oplus M$ the trivial extension of the DG-Lie algebra L by the L -module M , there exists a natural sequence of embeddings of DG-vector spaces $\text{Hom}_{\mathbb{K}}^*(L^{\wedge p}, M) \subset \text{Hom}_{\mathbb{K}}^*(K^{\wedge p}, K)$ commuting with the operators δ . Now, under the assumption $M = L$ the proof becomes tedious but completely straightforward, cf. also [4, 5, 29]. Alternatively one can observe that when ϕ, x_1, \dots, x_p have even degree the above definition of δ is, up to sign, the same of [16]; then one can use the standard trick (cf. [1, 31]) of taking the tensor products of L, M with a suitable Grassmann algebra in order to reduce the verification of $(\delta^2\phi)(x_1, \dots, x_p) = 0$ to the

case where ϕ, x_1, \dots, x_p have even degree. A more conceptual description of the Chevalley-Eilenberg complex will be given later as the und calage of the DG-vector space of coderivations of a differential symmetric coalgebra, cf. also [22, 29].

Notice that for $\phi \in \text{Hom}_{\mathbb{K}}^*(L, M)$, the condition $\delta\phi = 0$ can be written as

$$\phi([x, y]) = [\phi(x), y] - (-1)^{\bar{x}} \bar{y} [\phi(y), x] = [\phi(x), y] + (-1)^{\bar{x}} \bar{\phi}[x, \phi(y)]$$

and then the kernel of $\text{Hom}_{\mathbb{K}}^*(L, M) \xrightarrow{\delta} \text{Hom}_{\mathbb{K}}^*(L^{\wedge 2}, M)$ is the space of derivations $\phi: L \rightarrow M$. On the other side the image of $\delta: M \rightarrow \text{Hom}_{\mathbb{K}}^*(L, M)$ is, by definition, the space of inner derivations and then

$$H^1(CE(L, M), \delta) = \frac{\{\text{derivations } L \rightarrow M\}}{\{\text{inner derivations}\}}.$$

Similarly it is proved that $\bar{\delta}\delta + \delta\bar{\delta} = 0$, giving a double complex structure $(CE(L, M), \delta, \bar{\delta})$.

Definition 2.1. The Chevalley-Eilenberg cohomology $H_{CE}^*(L, M)$ of the differential graded Lie algebra L with coefficients in the L -module M is the cohomology of the total complex $\text{Tot}^{\Pi}(CE(L, M), \delta, \bar{\delta})$.

In other words, $H_{CE}^*(L, M)$ is the cohomology of the complex $\dots \rightarrow A^i \xrightarrow{\delta + \bar{\delta}} A^{i+1} \rightarrow \dots$, where

$$A^n = \prod_{p+q=n} \text{Hom}_{\mathbb{K}}^q(L^{\wedge p}, M).$$

The Chevalley-Eilenberg complex carries the natural, decreasing, exhaustive and complete filtration

$$F^p CE(L, M) = \text{Hom}_{\mathbb{K}}^* \left(\bigoplus_{i \geq p} \bigwedge^i L, M \right), \quad p \geq 0.$$

We shall denote by $(E(L, M)_r^{p,q}, d_r)$ the associated (Chevalley-Eilenberg) cohomology spectral sequence.

Example 2.2. If L, M have trivial differentials, then $H^*(L) = L$, $H^*(M) = M$ and therefore $E(L, M)_0^{p,q} = E(L, M)_1^{p,q}$. Moreover the spectral sequence degenerates at E_2 (i.e., $d_r = 0$ for every $r \geq 2$) and

$$H_{CE}^i(L, M) = \prod_{p \geq 0} E(L, M)_2^{p, i-p},$$

where

$$E(L, M)_2^{p, i-p} = \frac{\ker(\text{Hom}_{\mathbb{K}}^{i-p}(L^{\wedge p}, M) \xrightarrow{\delta} \text{Hom}_{\mathbb{K}}^{i-p}(L^{\wedge p+1}, M))}{\delta \text{Hom}_{\mathbb{K}}^{i-p}(L^{\wedge p-1}, M)}.$$

In general, since

$$E(L, M)_0^{p,*} = \frac{F^p CE(L, M)}{F^{p+1} CE(L, M)} = \text{Hom}_{\mathbb{K}}^*(L^{\wedge p}, M)$$

and the field \mathbb{K} is assumed to be of characteristic 0, we have $H^*(L^{\wedge p}) = H^*(L)^{\wedge p}$ (see e.g. [30, pag. 280]) and then

$$\begin{aligned} (2.1) \quad E(L, M)_1^{p,q} &= H^q(\text{Hom}_{\mathbb{K}}^*(L^{\wedge p}, M), \bar{\delta}) = \text{Hom}_{\mathbb{K}}^q(H^*(L)^{\wedge p}, H^*(M)) \\ &= E(H^*(L), H^*(M))_1^{p,q}. \end{aligned}$$

The differential $d_1: E(L, M)_1^{p,q} \rightarrow E(L, M)_1^{p+1,q}$ depends only by the graded Lie algebra $H^*(L)$ and its module $H^*(M)$, giving

$$E(L, M)_2^{p,*} = E(H^*(L), H^*(M))_2^{p,*} = H^p(CE(H^*(L), H^*(M)), \delta)$$

and therefore

$$E(L, M)_2^{1,*} = E(H^*(L), H^*(M))_2^{1,*} = \frac{\{\text{derivations } H^*(L) \rightarrow H^*(M)\}}{\{\text{inner derivations}\}}.$$

3. STATEMENT OF THE MAIN RESULTS

For the clarity of exposition, we list here the main results proved in this paper; the proofs rely on the theory of $L_\infty[1]$ -algebras and will be postponed in next sections.

Definition 3.1 (Euler class). The *Euler class* of a morphism of differential graded Lie algebras $f: L \rightarrow M$ is the element $e_f \in E(L, M)_2^{1,0} = E(H^*(L), H^*(M))_2^{1,0}$ corresponding to the Euler derivation

$$e_f: H^*(L) \rightarrow H^*(M), \quad e_f(x) = \deg(x) f(x).$$

The Euler class of a DG-Lie algebra L is defined as the Euler class of the identity on L .

Lemma 3.2. *Let $e \in E(L, L)_2^{1,0}$ be the Euler class of a differential graded Lie algebra L . If $d_2(e) = \dots = d_{r-1}(e) = 0$ for some $r > 2$, then $d_k = 0$ for every $2 \leq k < r$; in particular $E(L, L)_r^{p,q} = E(L, L)_2^{p,q}$.*

Every morphism of differential graded Lie algebras $f: L \rightarrow M$ induces by composition two natural morphisms of double complexes

$$CE(L, L) \xrightarrow{f_*} CE(L, M) \xleftarrow{f^*} CE(M, M)$$

and then also two morphisms of spectral sequences

$$(3.1) \quad E(L, L)_r^{p,q} \xrightarrow{f_*} E(L, M)_r^{p,q} \xleftarrow{f^*} E(M, M)_r^{p,q}.$$

preserving Euler classes. If f is a quasi-isomorphism, then by (2.1) the maps in (3.1) are isomorphisms for $r \geq 1$. Therefore, the truncation at $r \geq 1$ of the spectral sequence $E(L, L)_r^{p,q}$ and the Euler class are homotopy invariants of L .

Theorem 3.3 (Formality criterion). *Let $(E(L, L)_r^{p,q}, d_r)$ be the Chevalley-Eilenberg spectral sequence of a differential graded Lie algebra L . Then the following conditions are equivalent:*

- (1) L is formal;
- (2) the spectral sequence $E(L, L)_r^{p,q}$ degenerates at E_2 ;
- (3) denoting by

$$e \in E(L, L)_2^{1,0} = \frac{\text{Der}_{\mathbb{K}}^0(H^*(L), H^*(L))}{\{[x, -] \mid x \in H^0(L)\}}, \quad e(x) = \deg(x) \cdot x,$$

the Euler class of L , we have $d_r(e) = 0 \in E(L, L)_r^{r+1, 1-r}$ for every $r \geq 2$;

According to Lemma 3.2 the above item (3) makes sense. By the above considerations about the homotopy invariance of the Chevalley-Eilenberg spectral sequence and Euler classes, the only non trivial implication is (3 \Rightarrow 1).

Theorem 3.4 (Formality transfer). *Let $f: L \rightarrow M$ be a morphism of differential graded Lie algebras. Assume that*

- (1) M is formal;
- (2) for every $p \geq 3$ the map

$$f: E(H^*(L), H^*(L))_2^{p, 2-p} \rightarrow E(H^*(L), H^*(M))_2^{p, 2-p}$$

is injective.

Then also L is formal.

As we have already pointed out, the above Item (2) holds whenever the natural map

$$f: H_{CE}^2(H^*(L), H^*(L)) \rightarrow H_{CE}^2(H^*(L), H^*(M))$$

is injective.

Corollary 3.5. *Let L, M be a differential graded Lie algebra. Then $L \times M$ is formal if and only if both L and M are formal.*

Proof. Immediate consequence of Theorem 3.4, since L (resp.: M) is a direct summand of the L -module (resp.: M -module) $L \times M$. \square

The next corollary in the Lie analog of a remarkable result by Sullivan, Halperin and Stasheff [10, Cor. 6.9].

Corollary 3.6. *Let L be a differential graded Lie algebra and let A be a unitary differential graded commutative \mathbb{K} -algebra. If $H^*(A) \neq 0$ and $L \otimes A$ is formal, then also L is formal.*

Proof. Let's denote by $d: A^i \rightarrow A^{i+1}$ the differential of A , then $d(1) = 0$ and the assumption $H^*(A) \neq 0$ implies that the cohomology class of 1 is non trivial in $H^0(A)$: in fact, if $1 = da$ for some $a \in A^{-1}$, then for every $b \in A$ such that $d(b) = 0$ we have $d(ab) = b$. Thus the morphism $\mathbb{K} \rightarrow A, \alpha \mapsto \alpha 1$, is injective in cohomology and therefore there exists a direct sum decomposition $A = \mathbb{K} \oplus B$ with $d(B) \subseteq B$.

Now the proof follows from Theorem 3.4, since L is a direct summand of the L -module $L \otimes A = L \oplus (L \otimes B)$. \square

Definition 3.7 ([12]). A graded Lie algebra H is *intrinsically formal* if every differential graded Lie algebra L such that $H^*(L) \cong H$ is formal.

Putting $M = 0$ in Theorem 3.4 we recover the well known fact [12, 17] that a graded Lie algebra L with $E(L, L)_2^{p, 2-p} = 0$ for every $p \geq 3$ is intrinsically formal. After Theorem 3.3, another sufficient condition for intrinsic formality is given by the vanishing of the Euler class.

Corollary 3.8. *For every graded Lie algebra M and every $h \in M^0$ the graded Lie subalgebra*

$$H = \{x \in M \mid [h, x] = \deg(x) x\}$$

is intrinsically formal.

Proof. Notice first that $h \in H^0$ and then the Euler derivation $e = [h, -]: H \rightarrow H$ is an inner derivation. Let L be a differential graded Lie algebra with $H^*(L) = H$, then

$$E(L, L)_2^{1,0} = E(H, H)_2^{1,0} = \frac{\text{Der}_{\mathbb{K}}^0(H, H)}{\{[x, -] \mid x \in H^0\}},$$

and therefore the Euler class is trivial in $E(L, L)_2^{1,0}$. \square

Example 3.9. For every graded vector space V , the graded Lie algebras $\text{Hom}_{\mathbb{K}}^*(V, V)$, $\text{Hom}_{\mathbb{K}}^{\geq 0}(V, V)$ and $\text{Hom}_{\mathbb{K}}^{\leq 0}(V, V)$ are intrinsically formal. In fact, denoting by

$$h \in \text{Hom}_{\mathbb{K}}^0(V, V), \quad h(v) = \deg(v) v,$$

we have

$$[h, f] = e(f) = \deg(f) f, \quad \text{for every } f \in \text{Hom}_{\mathbb{K}}^*(V, V).$$

Example 3.10. For every graded commutative algebra A , the graded Lie algebra $\text{Der}_{\mathbb{K}}^*(A, A)$ is intrinsically formal. In fact, denoting by

$$h \in \text{Der}_{\mathbb{K}}^0(A, A), \quad h(v) = \deg(v)v,$$

we have

$$[h, f] = e(f) = \deg(f)f, \quad \text{for every } f \in \text{Der}_{\mathbb{K}}^*(A, A).$$

The same conclusion applies to every graded Lie subalgebra of $\text{Hom}_{\mathbb{K}}^*(A, A)$ containing the derivation h , e.g. the algebra of differential operators.

4. REVIEW OF $L_{\infty}[1]$ -ALGEBRAS AND NIJENHUIS-RICHARDSON BRACKET

Given a graded vector space V , the symmetric coalgebra generated by V is the graded vector space $S^c(V) = \bigoplus_{n \geq 0} V^{\odot n}$ equipped with the coproduct

$$\Delta(1) = 1 \otimes 1, \quad \Delta(v) = 1 \otimes v + v \otimes 1,$$

and more generally

$$\begin{aligned} \Delta: S^c(V) &\rightarrow S^c(V) \otimes S^c(V), \\ \Delta(v_1 \odot \cdots \odot v_n) &= \sum_{k=0}^n \sum_{\sigma \in S(n, n-k)} \epsilon(\sigma) (v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}) \otimes (v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(n)}) \end{aligned}$$

where $S(k, n-k)$ is the set of permutations of $\{1, \dots, n\}$ such that

$$\sigma(1) < \cdots < \sigma(k), \quad \sigma(k+1) < \cdots < \sigma(n),$$

and $\epsilon(\sigma)$ is the Koszul sign. For every positive integer m the subspace $\bigoplus_{0 \leq n \leq m} V^{\odot n}$ is a graded subcoalgebra of $S^c(V)$.

Throughout all this paper we shall use in force the following notation: whenever $i, j \geq 0$ and $f: S^c(V) \rightarrow S^c(W)$ is a linear map, we shall denote by $f_j^i: V^{\odot j} \rightarrow W^{\odot i}$ the composite map

$$V^{\odot j} \xrightarrow{\text{inclusion}} S^c(V) \xrightarrow{f} S^c(W) \xrightarrow{\text{projection}} W^{\odot i}.$$

The composition of f with the projection $S^c(W) \rightarrow W$ is called the *corestriction of f* ; equivalently the corestriction of f is the linear map $\sum_{j \geq 0} f_j^1$.

The projection $S^c(V) \rightarrow V^{\odot 0} = \mathbb{K}$ is a counity, while the inclusion $\mathbb{K} = V^{\odot 0} \rightarrow S^c(V)$ is an augmentation. With a little abuse of language, *by a morphism $f: S^c(V) \rightarrow S^c(W)$ of symmetric coalgebras we shall mean a morphism of graded augmented coalgebras*, i.e., a morphism of graded coalgebras such that $f(1) = 1$. In practice the assumption $f(1) = 1$ is equivalent to the fact that f is non trivial: it is an easy exercise to show that, given a morphism of graded coalgebras $f: S^c(V) \rightarrow S^c(W)$, then either $f = 0$ or $f(1) = 1$.

The following propositions are well known (see e.g. [21, 24]) and, in any case, easy to prove.

Proposition 4.1. *Every morphism of symmetric coalgebras $f: S^c(V) \rightarrow S^c(W)$ is uniquely determined by its corestriction, i.e., f is uniquely determined by the components f_j^1 , $j > 0$. Moreover, for every $n > 0$ and $v_1, \dots, v_n \in V$ we have*

$$f(v_1 \odot \cdots \odot v_n) - f_1^1(v_1) \odot \cdots \odot f_1^1(v_n) \in \bigoplus_{0 < i < n} W^{\odot i}.$$

In particular f is an isomorphism if and only if f_1^1 is an isomorphism.

For every graded coalgebra C we shall denote by $\text{Coder}^*(C, C)$ the graded Lie algebra of coderivations of C .

Proposition 4.2. *The corestriction map gives an isomorphism of graded vector spaces*

$$\text{Coder}^*(S^c(V), S^c(V)) \rightarrow \text{Hom}_{\mathbb{K}}^*(S^c(V), V) = \prod_{k \geq 0} \text{Hom}_{\mathbb{K}}^*(V^{\odot k}, V)$$

whose inverse map

$$\text{Hom}_{\mathbb{K}}^*(V^{\odot k}, V) \ni q \mapsto \widehat{q} \in \text{Coder}^*(S^c(V), S^c(V))$$

is described explicitly by the formulas

$$\widehat{q}(v_1 \odot \cdots \odot v_n) = \sum_{\sigma \in S(k, n-k)} \epsilon(\sigma) q(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(n)} .$$

For $k = 0$ the formula of Proposition 4.2 should be interpreted in the following sense: if $q \in \text{Hom}_{\mathbb{K}}^*(V^{\odot 0}, V) \cong \text{Hom}_{\mathbb{K}}^*(\mathbb{K}, V)$, then

$$\widehat{q}(1) = q(1), \quad \widehat{q}(v_1 \odot \cdots \odot v_n) = q(1) \odot v_1 \odot \cdots \odot v_n .$$

Therefore, for $q \in \text{Hom}_{\mathbb{K}}^*(V^{\odot k}, V)$ we have $\widehat{q}(V^{\odot n}) \subseteq V^{\odot n-k+1}$.

The graded commutator on $\text{Coder}^*(S^c(V), S^c(V))$ induces, via the corestriction isomorphism, a bracket

$$[-, -]_{NR}: \text{Hom}_{\mathbb{K}}^*(S^c(V), V) \times \text{Hom}_{\mathbb{K}}^*(S^c(V), V) \rightarrow \text{Hom}_{\mathbb{K}}^*(S^c(V), V) ,$$

known as Nijenhuis-Richardson bracket. In the notation of Proposition 4.2 we have

$$\begin{aligned} [-, -]_{NR}: \text{Hom}_{\mathbb{K}}^*(V^{\odot n}, V) \times \text{Hom}_{\mathbb{K}}^*(V^{\odot m}, V) &\rightarrow \text{Hom}_{\mathbb{K}}^*(V^{\odot n+m-1}, V) , \\ [f, g]_{NR} &= f \widehat{g} - (-1)^{\overline{f}} \widehat{g} f . \end{aligned}$$

Definition 4.3. An $L_{\infty}[1]$ -algebra is a graded vector space V equipped with a coderivation of degree $+1$, $Q \in \text{Coder}^1(S^c(V), S^c(V))$ such that $Q(1) = 0$ and $QQ = \frac{1}{2}[Q, Q] = 0$. An L_{∞} -morphism $f: (V, Q) \dashrightarrow (W, R)$ of $L_{\infty}[1]$ -algebras is a morphism of symmetric coalgebras $f: S^c(V) \rightarrow S^c(W)$ such that $fQ = Rf$.

Thus, see e.g. [19], there exists a canonical bijection between the set of L_{∞} -algebra structures of a graded vector space V and the set of $L_{\infty}[1]$ -algebra structures of a graded vector space $V[1]$. In particular every result about L_{∞} -algebras holds, mutatis mutandis, also for $L_{\infty}[1]$ -algebras.

Clearly we can define an $L_{\infty}[1]$ -algebra also in terms of the Nijenhuis-Richardson bracket: more precisely an $L_{\infty}[1]$ -algebra is an ∞ -uple (V, q_1, q_2, \dots) , where $q_n \in \text{Hom}_{\mathbb{K}}^1(V^{\odot n}, V)$ are such that for every $n > 0$

$$\sum_{k=1}^{n-1} [q_k, q_{n-k}]_{NR} = 0 ;$$

the relation between the above two definitions is given by

$$(V, Q) \mapsto (V, Q_1^1, Q_2^1, Q_3^1, \dots), \quad (V, q_1, q_2, \dots) \mapsto \left(V, \sum_{i \geq 0} \widehat{q}_i \right) .$$

Notice that, if (V, q_1, q_2, \dots) is an $L_{\infty}[1]$ -algebra we have $q_1 q_1 = 0$ and then (V, q_1) is a complex of vector spaces; we shall denote by $H^*(V)$ its cohomology, called the tangent cohomology of V . Since the equation $[q_1, q_2]_{NR} = 0$ may be written as

$$q_1(q_2(x, y)) + q_2(q_1(x), y) + (-1)^{\overline{q_2}} q_2(x, q_1(y)) = 0$$

we have that q_2 factors to a graded commutative (quadratic) bracket on tangent cohomology:

$$q_2: H^*(V) \times H^*(V) \rightarrow H^*(V) .$$

It is well known and in any case easy to prove that, for every L_∞ -morphism $f: (V, q_1, q_2, \dots) \dashrightarrow (W, r_1, r_2, \dots)$ its linear component

$$f_1^1: (V, q_1) \rightarrow (W, r_1)$$

is a morphism of complexes whose restriction to tangent cohomology

$$f_1^1: H^*(V) \rightarrow H^*(W)$$

commutes with the quadratic brackets q_2, r_2 .

Example 4.4 (Décalage). The décalage functor, from the category of differential graded Lie algebras to the category of $L_\infty[1]$ -algebra is defined as

$$(L, d, [-, -]) \mapsto (V, q_1, q_2, 0, 0, \dots)$$

where:

- (1) V is a graded vector space equipped with a linear map $s: V \rightarrow L$, of degree $+1$, inducing an isomorphism $s: V^i \xrightarrow{\cong} L^{i+1}$ for every i ;
- (2) the maps q_1, q_2 are defined by the formulas:

$$sq_1(v) = -d(sv), \quad sq_2(u, v) = -(-1)^{\overline{u}}[su, sv], \quad u, v \in V.$$

Conversely, every $L_\infty[1]$ -algebra (V, q_1, q_2, \dots) such that $q_i = 0$ for every $i \geq 3$ is the décalage of a differential graded Lie algebra.

Definition 4.5. An L_∞ -morphism $f: (V, q_1, q_2, \dots) \dashrightarrow (W, r_1, r_2, \dots)$ is called a weak equivalence if induces an isomorphism between tangent cohomology groups $f_1^1: H^*(V) \xrightarrow{\cong} H^*(W)$.

Definition 4.6. An $L_\infty[1]$ -algebra (V, q_1, q_2, \dots) is said to be minimal if $q_1 = 0$.

In other words, an $L_\infty[1]$ -algebra V is minimal if and only if $H^*(V) = V$.

Theorem 4.7 (Minimal model theorem). *For every $L_\infty[1]$ -algebra (V, q_1, q_2, \dots) there exist a minimal $L_\infty[1]$ -algebra $(W, 0, r_2, \dots)$ and two weak equivalences*

$$f: (V, q_1, q_2, \dots) \dashrightarrow (W, 0, r_2, \dots), \quad g: (W, 0, r_2, \dots) \dashrightarrow (V, q_1, q_2, \dots)$$

such that fg is the identity on W .

Proof. See e.g. either Lemma 4.9 of [19], or Theorem 3.0.9 of [20]. \square

The $L_\infty[1]$ -algebra $(W, 0, r_2, \dots)$ as in the above theorem is unique up to isomorphisms and it is called the *minimal model* of (V, q_1, q_2, \dots) . It is worth to mention that, as a consequence of Theorem 4.7, if $f: (V, q_1, q_2, \dots) \dashrightarrow (W, r_1, r_2, \dots)$ is a weak equivalence, then there exists a weak equivalence $g: (W, r_1, r_2, \dots) \dashrightarrow (V, q_1, q_2, \dots)$ such that $g_1^1: H^*(W) \rightarrow H^*(V)$ is the inverse of $f_1^1: H^*(V) \rightarrow H^*(W)$.

Definition 4.8. An $L_\infty[1]$ -algebra (V, q_1, q_2, \dots) is said to be formal if it is weak equivalent to a (pure quadratic) $L_\infty[1]$ -algebra $(W, 0, r_2, 0, 0, \dots)$. It is called homotopy abelian if it is weak equivalent to a trivial $L_\infty[1]$ -algebra $(W, 0, 0, 0, 0, \dots)$.

Therefore, an $L_\infty[1]$ -algebra (V, q_1, q_2, \dots) is formal if and only if its minimal model is isomorphic to $(H^*(V), 0, r_2, 0, 0, \dots)$, where r_2 is the restriction of the quadratic component q_2 to the tangent cohomology $H^*(V)$.

It is well known, see e.g. [11], that two differential graded Lie algebras are quasi-isomorphic if and only if they have isomorphic $L_\infty[1]$ minimal models. In particular a differential graded Lie algebra is formal (resp.: homotopy abelian) if and only if the associated $L_\infty[1]$ -algebra is formal (resp.: homotopy abelian).

5. HOMOTOPY INVARIANCE OF CHEVALLEY-EILENBERG SPECTRAL SEQUENCE

Let's recall, following [8], the detailed construction of the spectral sequence associated to a differential filtered complex.

Let M be an abelian group, equipped with a homomorphism $d: M \rightarrow M$ such that $d^2 = 0$ and a decreasing filtration $F^p M$, $p \in \mathbb{Z}$, such that $d(F^p M) \subset F^p M$. The associated spectral sequence (E_r^p, d_r) , $r \geq 0$, is defined as

$$Z_r^p = \{x \in F^p M \mid dx \in F^{p+r} M\}, \quad E_r^p = \frac{Z_r^p}{Z_{r-1}^{p+1} + dZ_{r-1}^{p-r+1}},$$

and the maps

$$d_r: E_r^p \rightarrow E_r^{p+r}$$

are induced by d in the obvious way. We have $d_r^2 = 0$ and there exist natural isomorphisms

$$E_{r+1}^p \simeq \frac{\ker(d_r: E_r^p \rightarrow E_r^{p+r})}{d_r(E_r^{p-r})}.$$

If $M = \oplus M^n$ is graded, $d(M^n) \subset M^{n+1}$ and every $F^p M = \oplus_n F^p M^n$ is a graded subgroup, then every group E_r^p inherits a natural graduation, namely:

$$Z_r^{p,q} = \{x \in F^p M^{p+q} \mid dx \in F^{p+r} M^{p+q+1}\}, \quad E_r^{p,q} = \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1} + dZ_{r-1}^{p-r+1,q+r-2}},$$

$$E_r^p = \bigoplus_q E_r^{p,q}, \quad d_r: E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}, \quad E_{r+1}^{p,q} \simeq \frac{\ker(d_r: E_r^{p,q} \rightarrow E_r^{p+r,q-r+1})}{d_r(E_r^{p-r,q+r-1})}.$$

It is convenient to introduce a refinement of the usual notion of degeneration of a spectral sequence [6].

Definition 5.1. We shall say that a cohomology spectral sequence $(E_r^{p,q}, d_r)$ degenerates at $E_k^{a,b}$ if the map $d_r: E_r^{a,b} \rightarrow E_r^{a+r,b-r+1}$ vanishes for every $r \geq k$. A spectral sequence $(E_r^{p,q}, d_r)$ degenerates at E_k if $d_r = 0$ for every $r \geq k$.

For every morphism of graded coalgebras $f: C \rightarrow D$ we shall denote by $\text{Coder}^*(C, D; f)$ the graded vector space of coderivations $\alpha: C \rightarrow D$, with the structure of D -comodule on C induced by the morphism f . When f is the identity we shall simply denote $\text{Coder}^*(C, C) = \text{Coder}^*(C, C; \text{Id}_C)$.

Definition 5.2. The Chevalley-Eilenberg complex of an L_∞ -morphism $f: (V, Q) \dashrightarrow (W, R)$ of $L_\infty[1]$ -algebras is the filtered differential complex

$$\begin{aligned} CE(V, W; f) &= \text{Coder}^*(S^c(V), S^c(W); f) \\ &= \{\alpha \in \text{Hom}_{\mathbb{K}}^*(S^c(V), S^c(W)) \mid \Delta\alpha = (\alpha \otimes f + f \otimes \alpha)\Delta\}, \end{aligned}$$

where:

- (1) the filtration is defined as

$$F^p CE(V, W; f) = \{\alpha \in \text{Coder}^*(S^c(V), S^c(W); f) \mid \alpha(V^{\odot i}) = 0, \quad \forall i < p\};$$

- (2) the differential d is defined by the formula

$$d\alpha = R\alpha - (-1)^{\bar{\alpha}}\alpha Q.$$

As in Proposition 4.2, the corestriction map gives an isomorphism of graded vector spaces

$$\text{Coder}^*(S^c(V), S^c(W); f) \rightarrow \text{Hom}_{\mathbb{K}}^*(S^c(V), W) = \prod_{k \geq 0} \text{Hom}_{\mathbb{K}}^*(V^{\odot k}, W),$$

although the inverse map $\alpha \mapsto \hat{\alpha}$ is now described explicitly by a more complicated formula, cf. [24]. However for our applications we only need the description of $\hat{\alpha}$ in some particular and easy cases, namely:

- (1) for $w \in W$ the associated coderivation $\hat{w} \in \text{Coder}^*(S^c(V), S^c(W); f)$ satisfies the equalities

$$\hat{w}(1) = w, \quad \hat{w}(v) = w \otimes f_1^1(v), \quad v \in V.$$

- (2) for $\alpha \in \text{Hom}_{\mathbb{K}}^*(V, W)$ the corresponding coderivation satisfies $\hat{\alpha}(1) = 0$, $\hat{\alpha}(v) = \alpha(v)$ and

$$\hat{\alpha}(v_1 \odot v_2) = \alpha(v_1) \odot f_1^1(v_2) + (-1)^{\overline{v_1} \overline{v_2}} \alpha(v_2) \odot f_1^1(v_1), \quad v_1, v_2 \in V.$$

The cohomology spectral sequence of the filtered differential complex $CE(V, W; f)$ defined above will be denoted by $(E(V, W; f)_r^{p,q}, d_r)$.

Lemma 5.3. *Let $(E(V, W; f)_r, d_r)$ be the Chevalley-Eilenberg spectral sequence of an L_∞ -morphism $f: (V, 0, q_2, q_3, \dots) \dashrightarrow (W, 0, r_2, r_3, \dots)$ of minimal $L_\infty[1]$ -algebras. Assume that for some integer $k \geq 3$ we have $q_3 = \dots = q_k = 0$ and $r_3 = \dots = r_k = 0$. Then $d_r = 0$ for every $2 \leq r < k$ and therefore*

$$E(V, W; f)_2 = E(V, W; f)_3 = \dots = E(V, W; f)_k.$$

Proof. By induction it is sufficient to prove that $d_{k-1} = 0$. We shall write $Q = Q_2 + Q'$ for the codifferential of $S^c(V)$, where

$$Q_2 = \hat{q}_2, \quad Q' = \sum_{i > k} \hat{q}_i, \quad Q_2(V^{\odot i}) \subset V^{\odot i-1}, \quad Q'(V^{\odot i}) \subset \oplus_{j \leq i-k} V^{\odot j}.$$

Similarly we write $R = R_2 + R'$, where

$$R_2 = \hat{r}_2, \quad R' = \sum_{i > k} \hat{r}_i, \quad R_2(W^{\odot i}) \subset W^{\odot i-1}, \quad R'(W^{\odot i}) \subset \oplus_{j \leq i-k} W^{\odot j}.$$

An element $v \in E(V, W; f)_{k-1}^p$ is represented by a linear map $\alpha \in \text{Hom}_{\mathbb{K}}^*(V^{\odot p}, W)$ such that $R\hat{\alpha} - (-1)^{\overline{\alpha}}\hat{\alpha}Q \in F^{p+k-1}\text{Coder}^*(S^c(V), S^c(W); f)$. Since $k \geq 3$ we must have $R_2\hat{\alpha} - (-1)^{\overline{\alpha}}\hat{\alpha}Q_2 = 0$,

$$R\hat{\alpha} - (-1)^{\overline{\alpha}}\hat{\alpha}Q = R'\hat{\alpha} - (-1)^{\overline{\alpha}}\hat{\alpha}Q' \in F^{p+k}\text{Coder}^*(S^c(V), S^c(W); f)$$

and this implies that $d_{k-1}(v) = 0$. □

Lemma 5.4. *Let $f: (V, Q) \dashrightarrow (W, R)$ and $g: (W, R) \dashrightarrow (U, S)$ be L_∞ -morphisms of $L_\infty[1]$ -algebras. Then the composition maps give two morphisms of filtered differential complexes*

$$g_*: \text{Coder}^*(S^c(V), S^c(W); f) \rightarrow \text{Coder}^*(S^c(V), S^c(U); gf),$$

$$f^*: \text{Coder}^*(S^c(W), S^c(U); g) \rightarrow \text{Coder}^*(S^c(V), S^c(U); gf).$$

Proof. Since $f(V^{\odot n}) \subset \oplus_{i \leq n} W^{\odot i}$ it is obvious that f^* and g_* preserve the filtrations. Now

$$g(d\alpha) = gR\alpha - (-1)^{\overline{\alpha}}\alpha Q = Sg\alpha - (-1)^{\overline{\alpha}}g\alpha Q = d(g\alpha)$$

$$(d\alpha)f = S\alpha f - (-1)^{\overline{\alpha}}\alpha Rf = S\alpha f - (-1)^{\overline{\alpha}}\alpha f Q = d(\alpha f).$$

□

Proposition 5.5. *In the situation of Lemma 5.4 the composition maps induce two morphisms of spectral sequences*

$$E(W, U; g)_r^{p,q} \xrightarrow{f^*} E(V, U; gf)_r^{p,q} \xleftarrow{g_*} E(V, W; f)_r^{p,q} :$$

- (1) *if f is a weak equivalence, then $f^*: E(W, U; g)_r^{p,q} \rightarrow E(V, U; gf)_r^{p,q}$ is an isomorphism for every $r \geq 1$;*
- (2) *if g is a weak equivalence, then $g_*: E(V, W; f)_r^{p,q} \rightarrow E(V, U; gf)_r^{p,q}$ is an isomorphism for every $r \geq 1$.*

Proof. For every integer p , the corestriction map gives an isomorphism

$$F^p CE(V, W; f) \simeq \prod_{n \geq p} \text{Hom}_{\mathbb{K}}^*(V^{\odot n}, W)$$

and therefore

$$E(V, W; f)_0^{p,q} = \text{Hom}_{\mathbb{K}}^{p+q}(V^{\odot p}, W) .$$

Given an element $\alpha \in \text{Hom}_{\mathbb{K}}^*(V^{\odot p}, W)$ we have $\alpha = \hat{\alpha}|_{V^{\odot p}}$, $\hat{\alpha}(V^{\odot i}) = 0$ for every $i < p$ and then

$$\begin{aligned} \hat{\alpha}Q(v_1 \odot \cdots \odot v_n) &= \alpha \left(\sum_{i=1}^n (-1)^{\overline{v_1} + \cdots + \overline{v_{i-1}}} v_1 \odot \cdots \odot Q_1^1(v_i) \odot \cdots \odot v_n \right) \\ R\hat{\alpha}(v_1 \odot \cdots \odot v_n) &= R_1^1 \alpha(v_1 \odot \cdots \odot v_n) , \\ d_0 \alpha &= R_1^1 \alpha - (-1)^{\overline{\alpha}} \alpha \left(\sum_{i=1}^n \text{Id}^{\odot i-1} \odot Q_1^1 \odot \text{Id}^{\odot n-i} \right) . \end{aligned}$$

In other terms d_0 is the standard differential in $\text{Hom}_{\mathbb{K}}^*(V^{\odot n}, W)$; by Künneth formula (cf. [30, pag. 280])

$$E(V, W; f)_1^{p,q} = \text{Hom}_{\mathbb{K}}^{p+q}(H^*(V)^{\odot p}, H^*(W)).$$

The conclusion of the proof is now clear. \square

Definition 5.6. The *Euler derivation* of an L_{∞} -morphism $f: V \dashrightarrow W$ is the element

$$e_f \in E(V, W; f)_1^{1,-1} = \text{Hom}_{\mathbb{K}}^0(H^*(V), H^*(W))$$

defined as

$$e_f(v) = (\overline{v} + 1)f_1^1(v), \quad v \in H^*(V) .$$

We are now ready to prove the main result of this section.

Theorem 5.7. *Let W be the minimal model of an $L_{\infty}[1]$ -algebra V . Then there exists a morphism of spectral sequences*

$$E(V, V)_r^{p,q} \rightarrow E(W, W)_r^{p,q}$$

which is an isomorphism for every $r \geq 1$ and preserves the Euler derivations.

Proof. By minimal model theorem there exist two weak equivalences

$$g: V \dashrightarrow W, \quad f: W \dashrightarrow V$$

such that gf is the identity on W . It now sufficient to consider the morphisms

$$E(V, V)_r^{p,q} \xrightarrow{f^*} E(W, V; f)_r^{p,q} \xrightarrow{g_*} E(W, W; gf)_r^{p,q} = E(W, W)_r^{p,q}$$

and apply Proposition 5.5. \square

Lemma 5.8. *Let $e_f \in E(V, W; f)_1^{1,-1}$ be the Euler derivation of an L_{∞} -morphism. Then $d_1(e_f) = 0$.*

Proof. By Proposition 5.5 it is not restrictive to assume both V and W minimal $L_\infty[1]$ -algebras, say $V = (V, 0, q_2, \dots)$ and $W = (W, 0, r_2, \dots)$. Let's give an explicit description of the two differentials

$$\begin{array}{ccccc} E(V, W; f)_1^{0,0} & \xrightarrow{d_1} & E(V, W; f)_1^{1,-1} & \xrightarrow{d_1} & E(V, W; f)_1^{2,-2} \\ \parallel & & \parallel & & \parallel \\ W^0 & & \text{Hom}_{\mathbb{K}}^0(V, W) & & \text{Hom}_{\mathbb{K}}^0(V^{\wedge 2}, W) \end{array}$$

Given $w \in W^0$ the associated coderivation $\hat{w} \in \text{Coder}^*(S^c(V), S^c(W); f)$ satisfies the equalities

$$\hat{w}(1) = w, \quad \hat{w}(v) = w \otimes f_1^1(v), \quad v \in V$$

and then,

$$(d_1 w)(v) = r_2(w, f_1^1(v)) \in W.$$

Similarly for $\alpha \in \text{Hom}_{\mathbb{K}}^0(V, W)$ the corresponding coderivation satisfies $\hat{\alpha}(v) = \alpha(v)$,

$$\hat{\alpha}(v_1 \odot v_2) = \alpha(v_1) \odot f_1^1(v_2) + (-1)^{\overline{v_1} \overline{v_2}} \alpha(v_2) \odot f_1^1(v_1) = \alpha(v_1) \odot f_1^1(v_2) + f_1^1(v_1) \odot \alpha(v_2),$$

and therefore

$$(d_1 \alpha)(v_1 \odot v_2) = r_2(\alpha(v_1), f_1^1(v_2)) + r_2(f_1^1(v_1), \alpha(v_2)) - \alpha(q_2(v_1, v_2)).$$

In particular

$$\begin{aligned} (d_1 e_f)(v_1 \odot v_2) &= \\ &= r_2((\overline{v_1} + 1)f_1^1(v_1), f_1^1(v_2)) + r_2(f_1^1(v_1), (\overline{v_2} + 1)f_1^1(v_2)) - (\overline{v_1} + \overline{v_2} + 2)f_1^1(q_2(v_1, v_2)) \\ &= 0. \end{aligned}$$

□

Definition 5.9. The *Euler class* of an L_∞ -morphism $f: V \dashrightarrow W$ is the element

$$e_f \in E(V, W; f)_2^{1,-1}$$

defined as the class of the Euler derivation modulus $d_1(W^0)$. The *Euler class* of an L_∞ -algebra is the Euler class of the identity.

It is plain from the above results that the Euler class of an $L_\infty[1]$ -algebra is invariant under weak equivalence.

6. FORMALITY CRITERIA FOR $L_\infty[1]$ -ALGEBRAS

Lemma 6.1. Let k be a positive integer and let (V, q_1, q_2, \dots) be an $L_\infty[1]$ -algebra such that $q_i = 0$ for every $i \neq k$. Then:

- (1) the spectral sequence $E(V, V)_r^{p,q}$ degenerates at E_k ;
- (2) if the spectral sequence $E(V, V)_r^{p,q}$ degenerates at $E_{k-1}^{1,-1}$ then also $q_k = 0$.

Proof. Via the identification $CE(V, V) \simeq \text{Hom}_{\mathbb{K}}^*(S^c(V), V)$ the differential of the complex becomes $d = [q_k, -]_{NR}$ and then $d(\text{Hom}_{\mathbb{K}}^*(V^{\odot n}, V)) \subset \text{Hom}_{\mathbb{K}}^*(V^{\odot n+k-1}, V)$. Therefore if $\alpha = \sum \alpha_i$, with $\alpha_i \in \text{Hom}_{\mathbb{K}}^*(V^{\odot i}, V)$ and $d\alpha \in F^{n+k}CE(V, V)$ then $d\alpha_i = 0$ for $i \leq n$; therefore, for every $n \in \mathbb{Z}$ and every $r \geq k$ we have

$$dZ_r^n \subset d(F^{n+1}CE(V, V)) \cap F^{n+r}CE(V, V) = dZ_{r-1}^{n+1}$$

and this implies $d_r = 0$.

As regards the second item, we assume $k > 1$, being the case $k = 1$ completely trivial. Considering the identity map $\text{Id}_V \in \text{Hom}_{\mathbb{K}}^0(V, V)$ as an element of $F^1 CE(V, V)$ we have:

$$q_k = \frac{1}{k-1} [q_k, \text{Id}_V]_{NR} \in F^k CE(V, V), \quad \text{Id}_V \in Z_{k-1}^{1, -1}.$$

If the spectral sequence degenerates at $E_{k-1}^{1, -1}$, then the class of q_k is trivial in $E_{k-1}^{k, 1-k}$; in particular

$$q_k \in d(F^2 CE(V, V)) + F^{k+1} CE(V, V)$$

and this implies $q_k = 0$. \square

Lemma 6.2. *Let $(V, 0, q_2, 0, \dots, 0, q_i, \dots)$ be a minimal $L_\infty[1]$ -algebra such that $q_j = 0$ for every $2 < j < i$ and some $2 < i$. Hence, by Lemma 5.3 we have $E(V, V)_2^{p, q} = E(V, V)_{i-1}^{p, q}$. Denoting by $e \in E(V, V)_2^{1, -1}$ its Euler class we have:*

- (1) $[q_2, q_i]_{NR} = 0$;
- (2) $d_r(e) = 0 \in E(V, V)_r^{r+1, -r}$ for every $1 \leq r < i-1$;
- (3) if $d_{i-1}(e) = 0 \in E(V, V)_{i-1}^{i, 1-i} = E(V, V)_2^{i, 1-i}$, then there exists $\alpha \in \text{Hom}_{\mathbb{K}}^0(V^{\odot i-1}, V)$ such that $q_i = [q_2, \alpha]_{NR}$.

Proof. The first part is clear since $\sum_j [q_j, q_{i+2-j}]_{NR} = 0$. The Euler class e is induced by the linear map

$$e \in \text{Hom}_{\mathbb{K}}^0(V, V), \quad e(v) = (\overline{v} + 1)v,$$

and for every $\beta \in \text{Hom}_{\mathbb{K}}^h(V^{\odot j}, V)$ we have

$$[\beta, e]_{NR} = (j - h - 1)\beta.$$

Therefore, setting $q = \sum q_j$, we have $[q_2, e]_{NR} = 0$,

$$[q, e]_{NR} = (i-2)q_i + (i-1)q_{i+1} + \dots \in F^i CE(V, V) \cap d(F^1 CE(V, V)).$$

In particular $[q, e]_{NR} \in Z_j^{j+1, -j}$ for every $j < i$ and this implies that $d_r(e) = 0 \in E(V, V)_r^{r+1, -r}$ for $1 \leq r < i-1$. If $d_{i-1}(e) = 0 \in E_{i-1}^{i, 1-i}$, then

$$(i-2)q_i + (i-1)q_{i+1} + \dots \in Z_{i-2}^{i+1, -i} + dZ_{i-2}^{2, -1}$$

and then there exists a sequence $\alpha_j \in \text{Hom}_{\mathbb{K}}^0(V^{\odot j}, V)$, $j \geq 2$, such that

$$[q, \sum \alpha_j]_{NR} - (i-2)q_i \in F^{i+1} CE(V, V)$$

this implies that $[q_2, \alpha_j]_{NR} = 0$ for $j < i-1$ and $[q_2, \alpha_{i-1}]_{NR} = (i-2)q_i$. In particular $\alpha = \frac{\alpha_{i-1}}{i-2}$ is the required element. \square

Theorem 6.3. *For a minimal $L_\infty[1]$ -algebra $(V, 0, q_2, q_3, \dots)$ with Euler class $e \in E(V, V)_2^{1, -1}$, the following conditions are equivalent:*

- (1) *there exists an L_∞ -isomorphism $f: (V, 0, q_2, 0, 0, \dots) \dashrightarrow (V, 0, q_2, q_3, \dots)$;*
- (2) *the spectral sequence $E(V, V)_r^{p, q}$ degenerates at E_2 ;*
- (3) *$d_r(e) = 0 \in E(V, V)_r^{r+1, -r}$ for every $r \geq 2$.*

Proof. By Lemma 6.1 and the homotopy invariance of the Chevalley-Eilenberg spectral sequence, we only need to prove $(3 \Rightarrow 1)$. If $q_i = 0$ for every $i > 2$ there is nothing to prove,

otherwise let $i \geq 3$ be the smallest integer such that $q_i \neq 0$; by Lemma 6.2 there exists an operator $\alpha \in \text{Hom}_{\mathbb{K}}^0(V^{\odot i-1}, V)$ such that $[q_2, \alpha]_{NR} = q_i$. Denoting by $\hat{\alpha} \in \text{Coder}^0(S^c(V), S^c(V))$ the corresponding pronilpotent coderivation, by $Q = \sum_j \hat{q}_j$ and by

$$R = e^{-\hat{\alpha}} Q e^{\hat{\alpha}} = e^{[\hat{\alpha}, -]}(Q) = Q + [\hat{\alpha}, Q] + \frac{1}{2}[\hat{\alpha}, [\hat{\alpha}, Q]] + \cdots$$

we have that $e^{\hat{\alpha}}: (V, R) \dashrightarrow (V, Q)$ is an L_{∞} -morphism. Denoting by r and $q = \sum q_i$ the corestrictions of R and Q , respectively, we have:

$$r = e^{[\alpha, -]_{NR}}(q) = q + [\alpha, q]_{NR} + \cdots \equiv q_2 + q_i - [q_2, \alpha]_{NR} \equiv q_2 \quad \left(\text{mod } \prod_{j>i} \text{Hom}_{\mathbb{K}}^1(V^{\odot j}, V) \right).$$

Therefore $r = q_2 + r_{i+1} + r_{i+2} + \cdots$ and then we have an L_{∞} -isomorphism

$$e^{\hat{\alpha}}: (V, q_2, 0, \dots, 0, r_{i+1}, \dots) \dashrightarrow (V, q_2, 0, \dots, 0, q_i, q_{i+1}, \dots).$$

Since $e^{\hat{\alpha}}$ is the identity on $V^{\odot j}$ for every $j \leq i-2$, we can repeat the procedure infinitely many times and take f as the infinite composition product of the above exponentials. \square

Corollary 6.4. *For a minimal $L_{\infty}[1]$ -algebra $(V, 0, q_2, q_3, \dots)$ the following conditions are equivalent:*

- (1) $q_i = 0$ for every i ;
- (2) the spectral sequence $E(V, V)_r^{p,q}$ degenerates at E_1 ;
- (3) the spectral sequence $E(V, V)_r^{p,q}$ degenerates at $E_1^{1,-1}$.

Proof. Immediate consequence of Theorem 6.3 and Lemma 6.1. \square

Remark 6.5. In [17] it is proved (in the framework of A_{∞} -algebras) that property (1) of Theorem 6.3 is equivalent to the vanishing of a certain cohomology class, called “Kaledin class” in [23]. For a minimal $L_{\infty}[1]$ -algebras $(V, 0, q_2, \dots)$, the Kaledin class may be defined in the following way: let t be a central formal indeterminate of degree 0; by extending the Nijenhuis-Richardson bracket to $CE(V, V)[[t]]$ in the obvious way, by homogeneity we have

$$[q(t), q(t)]_{NR} = 0, \quad \text{where } q(t) = q_2 + tq_3 + t^2q_4 + \cdots$$

and then $d(t) = [q(t), -]_{NR}$ is a differential on the $\mathbb{K}[[t]]$ -module $CE(V, V)[[t]]$.

Taking the formal derivative on the variable t we have

$$\partial_t q(t) = q_3 + 2tq_4 + \cdots, \quad [q(t), \partial_t q(t)]_{NR} = 0$$

and the cohomology class

$$[\partial_t q(t)] \in H^1(CE(V, V)[[t]], d(t))$$

is called the Kaledin class of the minimal $L_{\infty}[1]$ -algebra $(V, 0, q_2, \dots)$; it is supported at $t = 0$, since

$$t \partial_t q(t) = tq_3 + 2t^2q_4 + \cdots = [q(t), e]_{NR}$$

where e is Euler class. In particular, the Kaledin class vanishes whenever the multiplication map $H^1(CE(V, V)[[t]]) \xrightarrow{t} H^1(CE(V, V)[[t]])$ is injective. Notice that there exists a short exact sequence of complexes

$$0 \rightarrow (CE(V, V)[[t]], d(t)) \xrightarrow{t} (CE(V, V)[[t]], d(t)) \xrightarrow{t \mapsto 0} (CE(V, V), [q_2, -]_{NR}) \rightarrow 0$$

and it is an easy exercise on spectral sequences to prove that if $E(V, V)_r^{p,q}$ degenerates at $E_2^{a,-a}$ for every $a \geq 1$, then the map

$$H^0(CE(V, V)[[t]], d(t)) \xrightarrow{t \mapsto 0} H^0(CE(V, V), [q_2, -]_{NR})$$

is surjective.

The following two corollaries follow immediately from the above results together with Theorem 5.7.

Corollary 6.6. *For an $L_\infty[1]$ -algebra V with Euler class $e \in E(V, V)_2^{1,-1}$, the following conditions are equivalent:*

- (1) V is formal;
- (2) the spectral sequence $E(V, V)_r^{p,q}$ degenerates at E_2 ;
- (3) $d_r(e) = 0 \in E(V, V)_r^{r+1, -r}$ for every $r \geq 2$.

Corollary 6.7. *For an $L_\infty[1]$ -algebra V the following conditions are equivalent:*

- (1) V is homotopy abelian;
- (2) the spectral sequence $E(V, V)_r^{p,q}$ degenerates at E_1 ;
- (3) the spectral sequence $E(V, V)_r^{p,q}$ degenerates at $E_1^{1,-1}$.

The equivalence $(1 \Leftrightarrow 2)$ of Corollary 6.7, together some nice applications, has been recently proved by R. Bandiera [2] in a different way.

A well known result, which is very useful in deformation theory (see e.g. [13, 14, 15, 18]) is that if $f: V \dashrightarrow W$ is an L_∞ -morphism, W is homotopy abelian and $f_1^1: H^*(V) \rightarrow H^*(W)$ is injective, then also V is homotopy abelian. If W is formal, then the injectivity in tangent cohomology is not sufficient to ensure the formality of V . However, we have the following result, proved as a consequence of Theorem 6.3.

Theorem 6.8 (Formality transfer). *Let $f: V \dashrightarrow W$ be an L_∞ -morphism of $L_\infty[1]$ -algebras such that:*

- (1) W is formal;
- (2) the map $f_*: E(V, V)_2^{p, 1-p} \rightarrow E(V, W; f)_2^{p, 1-p}$ is injective for every $p \geq 3$.

Then also V is formal.

Proof. It is not restrictive to assume V minimal and W purely quadratic, say

$$f: (V, 0, q_2, q_3, \dots) \dashrightarrow (W, 0, r_2, 0, 0, \dots).$$

If $q_i \neq 0$ for some $i > 2$, let $k \geq 3$ be the smallest integer such that $q_k \neq 0$; thus $q_i = 0$ for every $2 < i < k$. According to Lemma 5.3 we have

$$E(V, V)_2^{p, 1-p} = E(V, V)_{k-1}^{p, 1-p}, \quad E(V, W; f)_2^{p, 1-p} = E(V, W; f)_{k-1}^{p, 1-p}$$

and then also $f_*: E(V, V)_{k-1}^{p, 1-p} \rightarrow E(V, W; f)_{k-1}^{p, 1-p}$ is injective for every $p \geq 3$. We have two morphisms of spectral sequences

$$E(V, V)_r^{p,q} \xrightarrow{f_*} E(V, W; f)_r^{p,q} \xleftarrow{f^*} E(W, W)_r^{p,q}$$

Denoting by e_V, e_W and e_f the Euler classes of V, W and f respectively we have $f_*(e_V) = e_f = f^*(e_W)$ and then for every $2 \leq i < k$ we have

$$f_*(d_i e_V) = d_i(f_* e_V) = d_i(f^* e_W) = f^*(d_i e_W) = 0 \in E(V, W; f)_i^{i+1, -i}$$

and then $d_i e_V = 0 \in E(V, V)_i^{i+1, -i}$ for every $2 \leq i < k$. The same argument used in the proof of Theorem 6.3 implies that, up to composition with an L_∞ -isomorphism of V which is the identity on $V^{\odot i}$, $i < k-1$, we can assume $q_k = 0$. Repeating this step, possibly infinitely many times for a sequence of increasing values of k , we prove the formality of V . \square

7. FORMALITY CRITERIA FOR DIFFERENTIAL GRADED LIE ALGEBRAS

By using the décalage isomorphism we can rewrite every formality result for $L_\infty[1]$ -algebras in the framework of differential graded Lie algebras. If V, L are graded vector spaces, then every bijective linear map $s: V \rightarrow L$ of degree $+1$ extends naturally to a sequence of linear isomorphisms

$$s: \operatorname{Hom}_{\mathbb{K}}^n(V^{\odot k}, V) \xrightarrow{\simeq} \operatorname{Hom}_{\mathbb{K}}^{n-k+1}(L^{\wedge k}, L), \quad n, k \in \mathbb{Z}, \quad k \geq 0,$$

defined by the formulas

$$(7.1) \quad (s\phi)(sv_1, \dots, sv_k) = (-1)^{\sum_i (k-i)\overline{v_i}} s\phi(v_1, \dots, v_k), \quad v_1, \dots, v_k \in V.$$

Assume now that L is a differential graded Lie algebra and $(V, q_1, q_2, 0, \dots)$ its décalage, defined as in Example 4.4. A straightforward computation shows that the bijective linear map of degree $+1$

$$s: CE(V, V) \rightarrow CE(L, L),$$

defined in (7.1) commutes, in the graded sense, with the differentials:

$$s\bar{\delta} + [q_1, s(-)]_{NR} = s\delta + [q_2, s(-)]_{NR} = 0.$$

In particular, s gives a bijective morphism of spectral sequences of degree $+1$.

$$s: E(V, V)_r^{p,q} \rightarrow E(L, L)_r^{p,q+1}$$

preserving the Euler classes.

After this, it is now clear that Lemma 3.2 follows from Lemma 5.3, Theorem 3.3 is a direct consequence of Corollary 6.6, while Theorem 3.4 follows from Theorem 6.8.

One of the possible limitations in the application of Theorem 3.3 is that in general, the filtration $F^p CE(L, L)$ is not bounded and the spectral sequence $E(L, L)_r^{p,q}$ is not regular; thus it may be useful to restate our results in terms of the spectral sequences of the quotient complexes $CE(L, L)/F^p CE(L, L)$.

Lemma 7.1. *Let $F^p M$, $p \in \mathbb{Z}$, be a decreasing filtration of a differential graded abelian group M . Denote by $E_r^{p,q}$ the associated spectral sequence and, for every integer b , by $E(l)_r^{p,q}$ the spectral sequence of the quotient filtered complex $M/F^l M$. Denoting by $\pi: M \rightarrow M/F^l M$ the projection, we have:*

- (1) *if $p < l$, then $\pi: E_r^p \rightarrow E(l)_r^p$ is injective;*
- (2) *if $p + r \leq l$, then $\pi: E_r^p \rightarrow E(l)_r^p$ is surjective;*
- (3) *for a fixed pair a, b of integers, the spectral sequences $E_r^{p,q}$ degenerates at $E_k^{a,b}$ if and only if $E(l)_r^{p,q}$ degenerates at $E(l)_k^{a,b}$ for every l .*

Proof. The first two properties are clear for $r = 0$; by induction we may assume $r > 0$ and items (1), (2) true for $r - 1$. We have a commutative diagram

$$\begin{array}{ccccc} E_{r-1}^{p-r+1} & \xrightarrow{d} & E_{r-1}^p & \xrightarrow{d} & E_{r-1}^{p+r-1} \\ \downarrow \pi' & & \downarrow \pi & & \downarrow \pi'' \\ E(l)_{r-1}^{p-r+1} & \xrightarrow{d} & E(l)_{r-1}^p & \xrightarrow{d} & E(l)_{r-1}^{p+r-1} \end{array}$$

If $p + r \leq l$, by induction the maps π', π are isomorphisms, π'' is injective and then also the induced map $E_r^p \rightarrow E(l)_r^p$ is an isomorphism. If $p < l$ and $p + r > l$, then $E(l)_{r-1}^{p+r-1} = 0$, π is injective and π' is an isomorphism; thus the induced map $E_r^p \rightarrow E(l)_r^p$ is injective.

As regards (3), the if part follows immediately from (2). Conversely, if $E_r^{p,q}$ degenerates at $E_k^{a,b}$, then for every l and every $r \geq k$ we have a commutative diagram:

$$\begin{array}{ccc} E_r^{a,b} & \xrightarrow{0} & E_r^{a+r,b-r+1} \\ \downarrow \pi' & & \downarrow \pi \\ E(l)_r^{a,b} & \xrightarrow{d_r} & E(l)_r^{a+r,b-r+1} \end{array}$$

If $a+r \geq b$ then $E(l)_r^{a+r,b-r+1} = 0$, while if $a+r < b$, then π' is surjective; in both cases $d_r = 0$. \square

Thus, putting together Theorem 3.3 and Lemma 7.1 we obtain the following proposition.

Proposition 7.2. *Let L be a differential graded Lie algebras. For every positive integer l let $\tau_{<l}E(L, L)_r^{p,q}$ be the spectral sequence of the complex*

$$\frac{CE(L, L)}{F^l CE(L, L)} = \prod_{p=0}^{l-1} \text{Hom}_{\mathbb{K}}^*(L^{\wedge p}, L) .$$

Then the following conditions are equivalent:

- (1) L is formal;
- (2) the spectral sequence $\tau_{<l}E(L, L)_r^{p,q}$ degenerates at E_2 for every l ;
- (3) the spectral sequence $\tau_{<l}E(L, L)_r^{p,q}$ degenerates at $E_2^{1,0}$ for every l ;
- (4) denoting by $e \in \tau_{<l}E(L, L)_2^{1,0}$ the Euler class of L , we have $d_r(e) = 0 \in \tau_{<l}E(L, L)_r^{r+1,1-r}$ for every $r \geq 2$; and every l ;

8. THE ROLE OF FORMAL DG-LIE ALGEBRAS IN DEFORMATION THEORY

The the role of differential graded Lie algebras in deformation theory was clear since the mid sixties when Nijenhuis and Richardson [27, 28] observed that many deformation problems are controlled by the Maurer-Cartan equation

$$dx + \frac{1}{2}[x, x] = 0, \quad x \in L^1,$$

for a suitable differential graded Lie algebra L . This point of view was extended by Deligne in terms of the philosophy that “in characteristic 0 every (infinitesimal) deformation problem is controlled by a differential graded Lie algebra, with quasi-isomorphic DG-Lie algebras giving the same deformation theory”, [9].

A more precise statement of the above philosophy can be stated in the framework of functors of Artin rings. Following [19], let \mathbb{K} be a field of characteristic 0, let \mathbf{Art} be the category of local Artin \mathbb{K} -algebras with residue field \mathbb{K} and let $F: \mathbf{Art} \rightarrow \mathbf{Set}$ be the functor of infinitesimal deformations of some “good” algebro-geometric structure. Then there exists a differential graded Lie algebras $(L, d, [-, -])$ such that $F \simeq \text{Def}_L$, where

$$\text{Def}_L(A) = \frac{\{x \in L^1 \otimes \mathfrak{m}_A \mid dx + \frac{1}{2}[x, x] = 0\}}{\text{gauge action of } \exp(L^0 \otimes \mathfrak{m}_A)},$$

\mathfrak{m}_A is the maximal ideal of A and the gauge action is defined by the formula

$$e^a * x := x + \sum_{n=0}^{\infty} \frac{[a, -]^n}{(n+1)!}([a, x] - da), \quad a \in L^0 \otimes \mathfrak{m}_A, \quad x \in L^1 \otimes \mathfrak{m}_A.$$

The basic theorem of deformation theory asserts that if $f: L \rightarrow M$ is a quasi-isomorphism of differential graded Lie algebras, then the induced natural transformation $f: \text{Def}_L \rightarrow \text{Def}_M$ is an isomorphism (see [25] and reference therein).

Proposition 8.1. *If a differential graded Lie algebra L is formal, then the two maps*

$$\text{Def}_L(\mathbb{K}[t]/(t^3)) \rightarrow \text{Def}_L(\mathbb{K}[t]/(t^2))$$

$$\text{Def}_L(\mathbb{K}[[t]]) := \lim_{\leftarrow n} \text{Def}_L(\mathbb{K}[t]/(t^n)) \rightarrow \text{Def}_L(\mathbb{K}[t]/(t^2))$$

have the same image.

Proof. Since Def_L is invariant under quasi-isomorphisms we may assume that L has trivial differential and therefore its Maurer-Cartan equation becomes $[x, x] = 0$, $x \in L^1$. Therefore $tx_1 \in \text{Def}_L(\mathbb{K}[t]/(t^2))$ lifts to $\text{Def}_L(\mathbb{K}[t]/(t^3))$ if and only if there exists $x_2 \in L^1$ such that

$$t^2[x_1, x_1] \equiv [tx_1 + t^2x_2, tx_1 + t^2x_2] \equiv 0 \pmod{t^3} \iff [x_1, x_1] = 0$$

and $[x_1, x_1] = 0$ implies that $tx_1 \in \text{Def}_L(\mathbb{K}[t]/(t^n))$ for every $n \geq 3$. \square

Notice that the formality of L does not imply that $\text{Def}_L(\mathbb{K}[[t]]) \rightarrow \text{Def}_L(\mathbb{K}[t]/(t^3))$ is surjective. The reader can easily verify that for a generic graded vector space V and $L = \text{Hom}_{\mathbb{K}}^*(V, V)$ the map $\text{Def}_L(\mathbb{K}[t]/(t^{n+1})) \rightarrow \text{Def}_L(\mathbb{K}[t]/(t^n))$ is not surjective for every $n \geq 2$.

The proof of Proposition 8.1 also implies that, when the deformation problem controlled by L admits a local moduli space \mathcal{M} and L is formal, then \mathcal{M} is defined by quadratic equations; more precisely \mathcal{M} is isomorphic to the germ at 0 of the quadratic cone defined by the equation $[x, x] = 0$, $x \in H^1(L)$; for a more detailed discussion and applications we refer to [9, 26].

Probably, the simplest example of local moduli space which is not defined by quadratic equations is given by the Hilbert scheme representing embedded deformations of the closed point inside the affine scheme $\text{Spec}(\mathbb{K}[x]/(x^3))$. By standard deformation theory, see e.g. [14], the construction of the differential graded Lie algebra L controlling this deformation problem is described by the following three steps:

- (1) replace the \mathbb{K} -algebra $\mathbb{K}[x]/(x^3)$ with a Koszul-Tate resolution, for instance with the DG-algebra

$$R = (\mathbb{K}[x, y], d), \quad \deg(x) = 0, \quad \deg(y) = -1, \quad d(y) = x^3,$$

where the closed point is the subscheme defined by the differential ideal $I = (x, y) \subset R$.

- (2) consider the differential graded Lie algebra $M = \text{Der}^*(R, R)$ and its subalgebra

$$N = \{\alpha \in M \mid \alpha(I) \subset I\}.$$

- (3) take L as the homotopy fiber of the inclusion $N \subset M$; as a concrete description of L we can take

$$L = \{a(t) \in M[t, dt] \mid a(0) = 0, a(1) \in N\}.$$

The non formality of L can also be checked algebraically, without relying on deformation theory. In fact M is the free $\mathbb{K}[x]$ -module generated by

$$y \frac{d}{dx}, \quad \frac{d}{dx}, \quad y \frac{d}{dy}, \quad \frac{d}{dy},$$

while N is the $\mathbb{K}[x]$ -submodule generated by

$$y \frac{d}{dx}, \quad x \frac{d}{dx}, \quad y \frac{d}{dy}, \quad x \frac{d}{dy}.$$

There exists a direct sum decomposition $M = N \oplus A$ where $A = A^0 \oplus A^1$ is the graded vector space generated by $u := \frac{d}{dx}$ and $\frac{d}{dy}$. Since A is an abelian graded Lie subalgebra of M and

$d = x^3 \frac{d}{dy} \in N$ we can apply Voronov's construction of higher derived brackets of an inner derivation [1, 31]. Denoting by $P: M \rightarrow A$ the projection with kernel N , the maps of degree +1:

$$q_n: A^{\odot n} \rightarrow A, \quad q_n(a_1, \dots, a_n) = P[\cdots [[d, a_1], a_2], \dots], \quad n \geq 1,$$

give an $L_\infty[1]$ structure on A which, according to [1, Thm. 1.3], is weak equivalent to the décalage of L . Since

$$[d, u] = \left[x^3 \frac{d}{dy}, \frac{d}{dx} \right] = -3x^2 \frac{d}{dy}, \quad [[d, u], u] = \left[-3x^2 \frac{d}{dy}, \frac{d}{dx} \right] = 6x \frac{d}{dy},$$

$$[[[d, u], u], u] = \left[6x \frac{d}{dy}, \frac{d}{dx} \right] = 6 \frac{d}{dy},$$

we have $q_1 = q_2 = 0$ and $q_3 \neq 0$. Thus the $L_\infty[1]$ -algebra $(A, q_1, q_2, q_3, \dots)$ is not formal.

Acknowledgment. I'm indebted with E. Arbarello for useful discussions and for pointing my attention on the paper [17].

REFERENCES

- [1] R. Bandiera: *Non-abelian higher derived brackets*. [arXiv:1304.4097](#) [math.QA]. [3](#), [20](#)
- [2] R. Bandiera: *Formality of Kapranov's brackets on pre-Lie algebras*. [arXiv:1307.8066](#) [math.QA]. [2](#), [16](#)
- [3] R. Bandiera, M. Manetti: *On coisotropic deformations of holomorphic submanifolds*. [arXiv:1301.6000v2](#) [math.AG]. [2](#)
- [4] D. Calaque and C. A. Rossi: *Lectures on Duflo isomorphisms in Lie algebra and complex geometry*. Ems Series of Lectures in Mathematics, European Mathematical Society (2011). [3](#)
- [5] C. Chevalley and S. Eilenberg: *Cohomology Theory of Lie Groups and Lie Algebras*. Transactions of the American Mathematical Society, Vol. **63**, No. 1 (1948), 85-124. [2](#), [3](#)
- [6] P. Deligne, P. Griffiths, J. Morgan and D. Sullivan: *Real homotopy theory of Kähler manifolds*. Invent. Math. **29** (1975) 245-274. [1](#), [10](#)
- [7] D. Fiorenza and M. Manetti: *Formality of Koszul brackets and deformations of holomorphic Poisson manifolds*. Homology, Homotopy and Applications, **14**, No. 2, (2012), 63-75; [arXiv:1109.4309v3](#) [math.QA]. [2](#)
- [8] R. Godement: *Topologie algébrique et théorie des faisceaux*. Hermann, Paris (1958). [10](#)
- [9] W.M. Goldman and J.J. Millson: *The deformation theory of representations of fundamental groups of compact Kähler manifolds* Publ. Math. I.H.E.S. **67** (1988) 43-96. [1](#), [18](#), [19](#)
- [10] S. Halperin and J. Stasheff: *Obstructions to homotopy equivalences*. Advances in Math. No. 32, 233-279 (1979). [6](#)
- [11] V. Hinich: *DG coalgebras as formal stacks*. Journal of Pure and Applied Algebra **162** (2001) 209-250; [arXiv:math/9812034v1](#) [math.AG]. [9](#)
- [12] V. Hinich: *Tamarkin's proof of Kontsevich formality theorem*. Forum Math. **15** (2003), no. 4, 591-614. [ArXiv:math/0003052](#). [6](#)
- [13] D. Iacono and M. Manetti: *An algebraic proof of Bogomolov-Tian-Todorov theorem*. In Deformation Spaces vol. **39**, Vieweg Verlag (2010), 113-133; [arXiv:0902.0732](#). [16](#)
- [14] D. Iacono and M. Manetti: *Semiregularity and obstructions of complete intersections*. Advances in Mathematics **235** (2013) 92-125; [arXiv:1112.0425](#) [math.AG]. [1](#), [16](#), [19](#)
- [15] D. Iacono: *Deformations and obstructions of pairs (X, D)* . [arXiv:1302.1149v5](#) [math.AG]. [2](#), [16](#)
- [16] N. Jacobson: *Lie algebras*. Wiley & Sons (1962). [3](#)
- [17] D. Kaledin: *Some remarks on formality in families*. Mosc. Math. J. **7** (2007) 643-652; [arXiv:math/0509699v4](#) [math.AG]. [2](#), [6](#), [15](#), [20](#)
- [18] L. Katzarkov, M. Kontsevich and T. Pantev: *Hodge theoretic aspects of mirror symmetry*. From Hodge theory to integrability and TQFT tt*-geometry, 87-174, Proc. Sympos. Pure Math., **78**, Amer. Math. Soc., Providence, (2008). [arXiv:0806.0107v1](#) [math.AG] [2](#), [16](#)
- [19] M. Kontsevich: *Deformation quantization of Poisson manifolds, I*. Letters in Mathematical Physics **66** (2003) 157-216; [arXiv:q-alg/9709040](#). [1](#), [8](#), [9](#), [18](#)
- [20] M. Kontsevich and Y. Soibelman: *Deformation theory, I*. Book in progress. Available from Y. Soibelman's web page. [9](#)

- [21] T. Lada and M. Markl: *Strongly homotopy Lie algebras*. Comm. Algebra **23** (1995) 2147-2161; [hep-th/9406095](#). [7](#)
- [22] A. Lazarev: *Models for classifying spaces and derived deformation theory*. [arXiv:1209.3866](#) [math.AT]. [4](#)
- [23] V. Lunts: *Formality of DG algebras (after Kaledin)*. Journal of Algebra, Volume **323**, Issue 4, (2010), 878-898. [arXiv:0712.0996](#) [math.AG]. [2](#), [15](#)
- [24] M. Manetti: *Lectures on deformations of complex manifolds*. Rend. Mat. Appl. (7) **24** (2004) 1-183; [arXiv:math.AG/0507286](#). [7](#), [11](#)
- [25] M. Manetti: *Differential graded Lie algebras and formal deformation theory*. In *Algebraic Geometry: Seattle 2005*. Proc. Sympos. Pure Math. **80** (2009) 785-810. [19](#)
- [26] E. Martinengo: *Local structure of Brill-Noether strata in the moduli space of flat stable bundles*. Rend. Semin. Mat. Univ. Padova **121** (2009), 259-280. [arXiv:0806.2056](#) [math.AG]. [19](#)
- [27] A. Nijenhuis and R. W. Richardson: *Cohomology and deformations of algebraic structures*. Bull. Amer. Math. Soc. Volume **70**, Number 3 (1964), 406-411. [18](#)
- [28] A. Nijenhuis and R. W. Richardson: *Deformation of Lie algebra structures*. J. Math. Mech. **17** (1967), 89-105. [18](#)
- [29] M. Penkava: *L-infinity algebras and their cohomology*. [arXiv:q-alg/9512014](#). [3](#), [4](#)
- [30] D. Quillen: *Rational homotopy theory*. Ann. of Math. **90** (1969) 205-295. [4](#), [12](#)
- [31] Th. Voronov: *Higher derived brackets for arbitrary derivations*. Travaux mathématiques, fasc. **XVI**, Univ. Luxemb., Luxembourg (2005), 163-186; [arXiv:0412202](#) [math.QA]. [3](#), [20](#)

UNIVERSITÀ DEGLI STUDI DI ROMA “LA SAPIENZA”,
 DIPARTIMENTO DI MATEMATICA “GUIDO CASTELNUOVO”,
 P.LE ALDO MORO 5, I-00185 ROMA, ITALY.

E-mail address: manetti@mat.uniroma1.it

URL: www.mat.uniroma1.it/people/manetti/